ABSTRACT HOMOTOPY. I

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1. *Introduction.*—Most theorems of homotopy theory, in particular those concerning homotopy and singular homology groups,[1] may be divided into two parts: (*a*) a theorem on abstract complexes and maps; (*b*) a "translation" of this abstract theorem into topological language by means of a singular functor (simplicial or cubical).

Such an abstract theorem, however, concerns only complexes which are the singular complex of a topological space. In this note it will be indicated how a homotopy theory may be developed for all abstract cubical complexes which satisfy only a certain *extension axiom;* homotopy groups will be introduced for all such complexes.

2. *Cubical Complexes.*[2]—The symbols ϵ and ω will always denote 0 or 1 even if indices are attached to them. A *cubical complex* K is a possibly void collection of elements σ (*cubes*) to each of which is attached a dimension $n \geq 0$ such that for each n-cube σ and integer i with $1 \leq i \leq n$ there are defined in K two $(n - 1)$-cubes $\sigma 0^i$ and $\sigma 1^i$ (*faces*) and for each n-cube σ and integer j with $1 \leq j \leq n + 1$ there is defined an $(n + 1)$-cube $\sigma\eta^j$ of K (*degenerate*), where the operators 0^i, 1^i and η^j satisfy the following identities (we recall that $\epsilon = 0, 1$ and $\omega = 0, 1$):

$$
\begin{aligned}
\epsilon^i\omega^{j-1} &= \omega^j\epsilon^i, && i < j,\\
\eta^{j-1}\eta^i &= \eta^i\eta^j, && i < j,\\
\eta^j\epsilon^i &= \epsilon^i\eta^{j-1}, && i < j,\\
&= \text{identity}, && i = j,\\
&= \epsilon^{i-1}\eta^j, && i > j,
\end{aligned}
\tag{1}
$$

A *subcomplex* M of K is a subcollection of K closed under the operators ϵ^i and η^j. A *cubical map* $f\colon K \to L$ is a dimension-preserving function which commutes with the operators ϵ^i and η^j. Let $\mathfrak{C}$ denote the resulting category.

A *tensor product* $K \otimes L$ can be defined which has a $(p + q)$-cube $\sigma \otimes \tau$ for every p-cube σ of K and q-cube τ of L, identifying the cubes $\sigma\eta^{p+1} \otimes \tau$ and $\sigma \otimes \tau\eta^1$. The operators ϵ^i and η^j are defined by

$$
\begin{aligned}
(\sigma \otimes \tau)\epsilon^i &= \sigma\epsilon^i \otimes \tau, & (\sigma \otimes \tau)\eta^j &= \sigma\eta^j \otimes \tau, && i, j \leq p,\\
(\sigma \otimes \tau)\epsilon^i &= \sigma \otimes \tau\epsilon^{i-p}, & (\sigma \otimes \tau)\eta^j &= \sigma \otimes \tau\eta^{j-p}, && i, j > p.
\end{aligned}
$$

Let I denote the cubical complex with a 1-cube ζ and two 0-cubes $\zeta_\epsilon = \zeta\epsilon^1$ as the only nondegenerate cubes; then two cubical maps $f_\epsilon\colon K \to L$ are called *homotopic* if there exists a cubical map $f_I\colon I \otimes K \to L$ (*homotopy*) such that $f_I(\zeta_\epsilon \otimes \sigma) = f_\epsilon\sigma$ for every cube σ of K (notation $f_I\colon f_0 \simeq f_1$ or $f_0 \simeq f_1$). This homotopy relation is reflexive but is not an equivalence relation.

A cubical complex K is called *connected* if for every two 0-cubes σ_ϵ of K there exists a 1-cube σ of K such that $\sigma\epsilon^1 = \sigma_\epsilon$.

Let $\partial\mathcal{G}$ be the category of chain complexes. Then, as usual,[3] a functor F^N: $\mathfrak{C} \to \partial\mathcal{G}$ can be defined as the quotient functor $F^N = F/F^D$, where F is obtained by regarding all cubes of a cubical complex as generators of a chain complex and F^D by considering the subcomplex generated by the degenerate cubes. The functor F^N preserves tensor products. As all (co-)homological notions have originally been defined on the category $\partial\mathcal{G}$, it follows that by composition with the functor F^N these notions also supply to the category $\mathcal{C}$, and, as F^N preserves homotopies, the resulting (co-)homology theories satisfy the homotopy axiom.

3. *Notational Conventions.*—Consider the symbols

$$\begin{bmatrix} \sigma_0{}^1, & \ldots\ldots\ldots, & \sigma_0{}^n \\ \sigma_1{}^1, & \ldots\ldots\ldots, & \sigma_1{}^n \end{bmatrix} \quad (2), \qquad \begin{bmatrix} \tau_0{}^1, & \ldots\ldots\ldots, & \tau_0{}^p \\ \tau_1{}^1, & \ldots\ldots\ldots, & \tau_1{}^p \end{bmatrix} \quad (2'),$$

$$\begin{vmatrix} \sigma_0{}^1, & \ldots\ldots\ldots, & \sigma_0{}^n \\ \sigma_1{}^1, & \ldots\ldots\ldots, & \sigma_1{}^n \end{vmatrix} \quad (3), \qquad \begin{vmatrix} \tau_0{}^1, & \ldots\ldots\ldots, & \tau_0{}^p \\ \tau_1{}^1, & \ldots\ldots\ldots, & \tau_1{}^p \end{vmatrix} \quad (3'),$$

$$\begin{bmatrix} \sigma_0{}^1, & \ldots, & \sigma_0{}^l, & \ldots, & \sigma_0{}^n \\ \sigma_1{}^1, & \ldots, & x_1{}^l, & \ldots, & \sigma_1{}^n \end{bmatrix} \quad (4), \qquad \begin{bmatrix} \tau_0{}^1, & \ldots, & \tau_0{}^l, & \ldots, & \tau_0{}^p \\ \tau_1{}^1, & \ldots, & x_1{}^l, & \ldots, & \tau_1{}^p \end{bmatrix} \quad (4'),$$

where $\sigma_\epsilon{}^i$ and $\tau_\epsilon{}^i$ are $(n-1)$-cubes of a cubical complex K, and p is an integer with $1 \leq p \leq n$.

Symbol (2) will denote the existence of an n-cube σ of K such that $\sigma\epsilon^i = \sigma_\epsilon{}^i$ for all ϵ^i. Then σ is called a *solvent* of (2), and (2) is called the *boundary* of σ. Symbol (3) will denote an arbitrary but fixed solvent of (2). Symbol (4) is called an *equation* in an $(n-1)$-cube $x_1{}^l$ of K and will mean that $\sigma_\epsilon{}^i\omega^{j-i} = \sigma_\omega{}^j\epsilon^i$ for $i < j$ and $\epsilon^i, \omega^j \neq 1^l$. It is called *solvable* if there exists an $(n-1)$-cube $\sigma_1{}^l$ of K (a *solution*) such that (2) holds or, equivalently, if there exists an n-cube σ of K (a *solvent* of (4)) such that $\sigma\epsilon^i = \sigma_\epsilon{}^i$ for $\epsilon^i \neq 1^l$, for then $\sigma 1^l$ is a solution. The definition of an equation in an $(n-1)$-cube $x_0{}^l$ of K is similar.

Symbol (2′) will mean that (2) holds, where $\sigma_\epsilon{}^i = \tau_\epsilon{}^i$ for $i \leq p$ and $\sigma_\epsilon{}^i = \tau_0{}^1\eta^1\epsilon^i = \tau_1{}^1\eta^1\epsilon^i$ for $i > p$. An n-cube σ of K is called a solvent of (2′) if it is a solvent of (2). We call (2′) an (*abbreviated*) *boundary* of σ. An arbitrary but fixed solvent of (2′) will also be denoted by (3′). If $p = 1$, then we often write $[\tau_0{}^1, \tau_1{}^1]$ and $|\tau_0{}^1, \tau_1{}^1|$ instead of (2′) and (3′). Likewise, the symbol (4′), an (*abbreviated*) *equation*, will mean that (4) holds, where $\sigma_\epsilon{}^i = \tau_\epsilon{}^i$ for $i \leq p$, $\epsilon^i \neq 1^l$ and $\sigma_\epsilon{}^i = \tau_0{}^1\eta^1\epsilon^i$ for $i > p$. It is called *solvable* if equation (4) is so or, equivalently, if there exists an $(n-1)$-cube $\tau_1{}^l$ of K (*solution*) such that (2′) holds. The usefulness of these abbreviations may be seen from the following theorem.

THEOREM 1. *Let $\tau_\epsilon{}^i$ ($i = 1, \ldots, p$ and $\epsilon^i \neq 1^l$) be $2p - 1$ $(n-1)$-cubes of $K \in \mathcal{C}$ such that*

$$\tau_\epsilon{}^i\omega^{j-1} = \tau_\omega{}^j\epsilon^i, \qquad i < j,$$

$$\tau_\epsilon{}^i \textit{ is a solvent of } \begin{bmatrix} \tau_\epsilon{}^i0^1, & \ldots, & \tau_\epsilon{}^i0^{p-1} \\ \tau_\epsilon{}^i1^1, & \ldots, & \tau_\epsilon{}^i1^{p-1} \end{bmatrix}.$$

Then (4′) holds. If, in addition, this abbreviated equation (4′) is solvable, then every solution has

$$\begin{bmatrix} \tau_0{}^11^{l-1}, & \ldots, & \tau_0{}^{l-1}1^{l-1}, & \tau_0{}^{l+1}1^l, & \ldots, & \tau_0{}^p1^l \\ \tau_1{}^11^{l-1}, & \ldots, & \tau_1{}^{l-1}1^{l-1}, & \tau_1{}^{l+1}1^l, & \ldots, & \tau_1{}^p1^l \end{bmatrix} \qquad (5)$$

as an abbreviated boundary, i.e. (5) holds.

4. *The Extension Axiom.*—We shall need the following properties of a cubical complex K:

a) Property $E\ (n, \epsilon, i)$: Every equation in an $(n - 1)$-cube $x_\epsilon{}^i$ of K is solvable.

b) Property HE (homotopy extension): Let $f_0: K \to L$ be a cubical map, let M be a subcomplex of K, and let g_I: $g_0 \simeq g_1$, where $g_0 = f_0 | M$. Then there exists a homotopy f_I: $f_0 \simeq f_1$ such that $f_I | I \otimes M = g_I$.

The strength of these properties may be indicated by the following theorems.

THEOREM 2. *A cubical complex K has the property HE if and only if it has the property $E\ (n, 1, 1)$ for all n.*

THEOREM 3. *If a cubical complex K has the property $E\ (n, 1, 1)$ for all n, then the homotopy relation $\simeq$ is an equivalence relation on the set of the cubical maps $L \to K$.*

THEOREM 4. *If a cubical complex K has the property $E\ (n, 0, n)$ for all n, then for every two q-cubes σ_ϵ of K there exists a $(q + 1)$-cube σ such that $\sigma\epsilon^1 = \sigma_\epsilon$.*

We now define an *E-complex* as a cubical complex K which satisfies the following axiom:

Extension axiom: K has the properties $E\ (n, 1, 1)$ and $E\ (n, 0, n)$ for all n.

The full subcategory of $\mathfrak{C}$ generated by the E-complexes will be denoted by $\mathfrak{C}_E$. A map of $\mathfrak{C}_E$ is called an *E-map*.

THEOREM 5. *An E-complex K has the property $E\ (n, \epsilon, i)$ for all values of n, ϵ and i, i.e., every equation of K is solvable.*

5. *The Solutions of an Equation.*—Two n-cubes σ_ϵ of a cubical complex K are called *compatible*[4] if their faces coincide, i.e., if $\sigma_0\epsilon^i = \sigma_1\epsilon^i$ for all ϵ^i. They are called *compatible and homotopic*[4] if $[\sigma_0, \sigma_1]$, i.e., if there exists an $(n + 1)$-cube σ of K such that $\sigma\epsilon^1 = \sigma_\epsilon$ and $\sigma\epsilon^i = \sigma_0\eta^1\epsilon^i$ for $i > 1$ (notation $\sigma_0 \sim \sigma_1$). This relation $\sim$ on the cubes of K is reflexive but need not be an equivalence relation. However, the following theorem holds.

THEOREM 6. *Let K be an E-complex. Then*

a) The relation $\sim$ is an equivalence relation on the cubes of K,

b) The set of the solutions of an equation of K is exactly an equivalence class of cubes of K, and

c) This class depends only on the equivalence classes of the cubes $\sigma_\epsilon{}^i$ of the equation.

The importance of this theorem lies in the fact that now symbols (2), (2′), (4), and (4′) may be considered as the definition of a kind of multimultiplication for the equivalence classes of n-cubes of an E-complex and that therefore these symbols still have a clear meaning if some of the entries are classes of n-cubes.

6. *The Homotopy Groups.*—Let ψ be an $(n$-$1)$-cube of an E-complex K. According to Theorem 6 the relation $\sim$ divides the solvents of $[\psi, \psi]$ into classes a, b, c, etc. Let 0 denote the class which contains $\psi\eta^1$. We now define the sum $a + b$ of two classes by the condition

$$\begin{bmatrix} a, & 0 \\ a + b, & b \end{bmatrix}. \tag{6}$$

It follows from Theorem 6 that the sum $a + b$ is uniquely determined for every a and b.

THEOREM 7. *The classes of the solvents of* $[\psi, \psi]$ *form a group* $\pi_n(K; \psi)$ *under the above definition of sum, the n-th homotopy group of K rel. the base-*$(n - 1)$*-cube* ψ.

Proof: Let α, β, γ, $\alpha + \beta$, etc., be cubes of the classes a, b, c, $a + b$, etc. It follows from Theorem 5 that both equations

$$\begin{bmatrix} x_0{}^1, & \psi\eta^1 \\ \alpha, & \beta \end{bmatrix} \quad (7), \qquad\qquad \begin{bmatrix} \beta, & \psi\eta^1 \\ \alpha, & x_1{}^2 \end{bmatrix} \quad (7'),$$

are solvable, and therefore, in view of Theorem 6, the same applies to the equations $x + b = a$ and $b + y = a$. It now remains to prove that the associative law holds. As $\alpha\eta^1\epsilon^1 = \alpha$, $\alpha\eta^1\epsilon^2 = \alpha\epsilon^1\eta^1 = \psi\eta^1$, and $\alpha\eta^1\epsilon^i = \alpha\eta^1\epsilon^i$, it follows that, for $\alpha = \beta$, $\psi\eta^i$ is a solution of (7′) (i.e. 0 is a right zero). Now application of Theorem 1 to

$$\begin{bmatrix} \begin{vmatrix} \alpha, & \psi\eta^1 \\ \alpha + \beta, & \beta \end{vmatrix} & \begin{vmatrix} \alpha, & \psi\eta^1 \\ \alpha, & \psi\eta^1 \end{vmatrix} & \begin{vmatrix} \psi\eta^1, & \psi\eta^1 \\ \psi\eta^1, & \psi\eta^1 \end{vmatrix} \\ x_1{}^1 & \begin{vmatrix} \alpha + \beta, & \psi\eta^1 \\ (\alpha + \beta) + \gamma, & \gamma \end{vmatrix} & \begin{vmatrix} \beta, & \psi\eta^1 \\ \beta + \gamma, & \gamma \end{vmatrix} \end{bmatrix} \text{ yields } \begin{bmatrix} \alpha, & \psi\eta^1 \\ (\alpha + \beta) + \gamma, & \beta + \gamma \end{bmatrix},$$

i.e., $a + (b + c) = (a + b) + c$, *Q.E.D.*

It immediately follows from the definitions that an E-map $f: K \to L$ induces homomorphisms $f_*: \pi_n(K; \psi) \to \pi_n(L; f\psi)$ for all n. The isomorphisms of the homotopy groups of a topological space induced by a path generalize to isomorphisms $\gamma_*: \pi_n(K; \gamma 1^1) \to \pi_n(K; \gamma 0^1)$ induced by an n-cube γ of K by the condition

$$\begin{bmatrix} \gamma, & \gamma_* a \\ \gamma, & a \end{bmatrix}, \tag{8}$$

where $a \in \pi_n(K; \gamma 1^1)$. Application of Theorem 4 then yields

THEOREM 8. *Let* ψ, χ *be* $(n - 1)$*-cubes of a connected E-complex K. Then there exists a not necessarily unique isomorphism* $\pi_n(K; \psi) \cong \pi_n(K; \chi)$.

The following can also be proved:

THEOREM 9. $\pi_n(K; \psi)$ *is Abelian for* $n > 1$.

For a set $\{\sigma\}$ of $2n + 2$ n-cubes $\sigma_\epsilon{}^i$ $(i = 1, \ldots, n + 1)$ of an E-complex K with $\sigma_\epsilon{}^i\omega^{j-1} = \sigma_\omega{}^j\epsilon^i$ for $i < j$, let $\alpha_0{}^1$ denote a solution of the equation in an n-cube $x_0{}^1$ of K involving the $\sigma_\epsilon{}^i$ except $\sigma_0{}^1$. We now define an element $c'\{\sigma\} \in \pi_n(K; \sigma_0{}^1 0^1)$ by the condition

$$\begin{bmatrix} \sigma_0{}^1, & c'\{\sigma\} \\ \alpha_0{}^1, & \sigma_0{}^1 1^1 \eta^1 \end{bmatrix}. \tag{9}$$

Using this and similar notions, the homotopy addition theorems[5] can be formulated and proved.

7. *Duality.*—Let $D: \mathcal{C} \to \mathcal{C}$ be the functor such that the cubical complex DK contains exactly one n-cube σ^* for every n-cube σ of K with operators $\sigma^* 0^i = (\sigma 1^{n+1-i})^*$, $\sigma^* 1^i = (\sigma 0^{n+1-i})^*$, and $\sigma^*\eta^j = (\sigma\eta^{n+2-j})^*$, and that for every cubical map $f: K \to L$ the map Df is determined by $(Df)\sigma^* = (f\sigma)^*$. Then, clearly, $DD = E$, the identity functor of $\mathcal{C}$. It follows easily that a cubical complex K has the property E (n, ϵ, i) if and only if DK has the property E $(n, 1 - \epsilon, n + 1 - i)$. Consequently, the extension axiom is self-dual, i.e., K is an E-complex if and only if DK is so.

Dualizing the homotopy relation, we call two cubical maps $f_\epsilon: K \rightarrow L$ *aft-homotopic* ($f_0 \approx f_1$) if $Df_1 \simeq Df_0$. In general, the relations $\simeq$ and $\approx$ do not coincide. However, we have

THEOREM 10. *Both homotopy relations coincide on* $\mathcal{C}_K$.

[1] See, for example, Eilenberg and Mac Lane, *Ann. Math.*, **51**, 514–533, 1950, and Serre, *Ann. Math.*, **54**, 425–505, 1951.

[2] This is the cubical analogue of the complete semi-simplicial complexes of Eilenberg and Zilber, *Ann. Math.*, **51**, 499–513, 1950.

[3] Cf. Eilenberg and Mac Lane, *Am. J. Math.*, **75**, 189–199, 1953.

[4] Cf. Eilenberg and Zilber, *op. cit.*

[5] Sze-tsen Hu, *Ann. Math.*, **58**, 108–122, 1953.

ON THE IRREDUCIBLE REPRESENTATIONS OF THE SYMMETRIC GROUP

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We have pointed out[1] in a recent note in these PROCEEDINGS that all representations of S_n, the symmetric group on n symbols, are linear combinations with integral coefficients of appropriately symmetrized Kronecker powers of the irreducible representation $\Gamma(n - 1, 1)$, of dimension $n - 1$, of S_n and have furnished these linear combinations for the irreducible representations $\Gamma(n - p, \lambda_2, \ldots)$ of S_n in the special cases $p = 2, 3, 4$. In an accompanying note[2] we have given certain rules which facilitate (and in important instances furnish) the analysis of the Kronecker product of two irreducible representations of S_n into its irreducible components. The problem of determining this analysis is closely related to that of analyzing the various appropriately symmetrized powers of $\Gamma(n - 1, 1)$, and the object of the present note is to indicate this connection and to extend the rules furnished in the note[2] just referred to. We also furnish the analysis of the various symmetrized powers $\Gamma(n - 1, 1) \otimes \{\mu\}$, (μ) a partition of p, of $\Gamma(n - 1, 1)$ in the cases $p = 5$ and $p = 6$.

With each partition $(\lambda) = (\lambda_1, \ldots, \lambda_K)$, $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k$, of n there is associated an irreducible representation $\Gamma(\lambda)$ of S_n, and, in the Kronecker product $\Gamma(\lambda) \times \Gamma(\lambda')$ of two such irreducible representations, the passage from the partition (λ) to the associated partition (λ^*) of n acts like a change of sign of a factor in the product of two real or complex numbers (the product of $\Gamma(\lambda')$ by $\Gamma(\lambda^*)$ being the associate of the product of $\Gamma(\lambda')$ by $\Gamma(\lambda)$). On writing $\lambda_1 = n - p$, $p = 0, 1, \ldots, n - 1$, (λ) appears as $(n - p, (\mu))$, where $(\mu) = (\lambda_2, \ldots, \lambda_k)$ is a partition of p, such that $n - p \geqslant \lambda_2$, and so we may regard the various irreducible representations of S_n as arranged in shells of varying depths, the number of representations in the shell of depth p being the number of those partitions of p whose first element $\leqslant n - p$. We may select from any pair of associated representations of S_n either of the two representations (the other being rejected as unessential), and we agree to select the representation of lesser depth. The effect of this selection is to reduce